

Estimates in Generalized Morrey Spaces for Weak Solutions to Divergence Degenerate Parabolic Systems *

Yan DONG, Maochun ZHU, Pengcheng NIU [†]

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Abstract: Let $X = (X_1, \dots, X_q)$ be a family of real smooth vector fields satisfying Hörmander's condition. The purpose of this paper is to establish gradient estimates in generalized Morrey spaces for weak solutions of the divergence degenerate parabolic system related to X :

$$u_t^i + X_\alpha^*(a_{ij}^{\alpha\beta}(z)X_\beta u^j) = g_i + X_\alpha^* f_i^\alpha(z),$$

where $\alpha, \beta = 1, 2, \dots, q$, $i, j = 1, 2, \dots, N$, X_α^* is the transposed vector field of X_α , $z = (t, x) \in \mathbb{R}^{n+1}$, and coefficients $a_{ij}^{\alpha\beta}(z)$ belong to the space VMO induced by the vector fields X_1, \dots, X_q .

Key words: divergence degenerate parabolic system; weak solution; Hörmander's vector fields; VMO function; generalized Morrey space.

1 Introduction

Let us consider a family of real smooth vector fields

$$X_j = \sum_{k=1}^n b_{jk}(x) \frac{\partial}{\partial x_k}, j = 1, 2, \dots, q, (q \leq n),$$

defined in a neighborhood $\tilde{\Omega}$ of some bounded domain $\Omega \subset \mathbb{R}^n$, satisfying Hörmander's condition, namely, the Lie algebra generated by the family $X = (X_1, \dots, X_q)$ at any point of $\tilde{\Omega}$ spans \mathbb{R}^n , see [10].

Equations and systems involving vector fields have received much attention during the recent years, see [1, 8, 12, 14, 15, 16, 19, 20, 21] etc.. The Morrey estimates for elliptic systems in Carnot-Carathéodory space have been studied by G. Di Fazio and M. Fanciullo in [6]. The aim of this paper is to establish

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[†]Corresponding author, E-mail addresses: pengchengniu@nwpu.edu.cn

gradient estimates in generalized Morrey spaces for weak solutions of the divergence degenerate parabolic system related to X . Of course, our work is greatly influenced by those in the classic Euclidean case, that is $X_i = \frac{\partial}{\partial x_i}$, $i = 1, \dots, n$, where fairly complete results have been obtained, for example, see [2, 9, 11, 18] etc.. For parabolic system with constant coefficients, Schauder and L^p estimates were studied by Schlag in [17], while when coefficients are discontinuous and belong some VMO space, McBride in [13] derived the generalized Morrey estimates for gradients of weak solutions. For some earlier studies, we quote [3, 4, 18] and the references therein.

In this paper, the degenerate parabolic system we considered is of the type

$$u_t^i + X_\alpha^*(a_{ij}^{\alpha\beta}(z)X_\beta u^j) = g_i + X_\alpha^* f_i^\alpha(z), \quad (1.1)$$

where $\alpha, \beta = 1, 2, \dots, q$; $i, j = 1, 2, \dots, N$, $z = (x, t) \in \mathbb{R}^{n+1}$, X_j^* is the transposed vector field of X_j , $X_j^* = -X_j + c_j$ ($c_j = -\sum_{k=1}^n \frac{\partial b_{jk}}{\partial x_k} \in C^\infty(\Omega)$), Ω is a bounded domain in \mathbb{R}^n .

The main difficulty in our setting is that the presence of commutators of vector fields which does not allow us to differentiate the equation. In order to overcome this and apply the method in [13] to our system, we need to resort to some conclusions proved by Xu in [20], and prove that some relative results in the classic Euclidean case are still hold in our setting.

Our basic assumption is:

(H) Let g_i and f_i^α in (1.1) belong to the generalized Morrey space $L_\varphi^{2,\lambda}(Q_T)$, $0 \leq \lambda < Q + 2$ (the number Q is the homogeneous dimension relative to Ω), and coefficients $a_{ij}^{\alpha\beta}(z)$ belong to $L^\infty(Q_T) \cap VMO(Q_T)$, where we refer to Section 2 for the precise meaning of $L_\varphi^{2,\lambda}(Q_T)$, Q_T , φ and $VMO(Q_T)$. Also let $a_{ij}^{\alpha\beta}(x, t)$ satisfy the uniform ellipticity condition:

$$\Lambda^{-1}|\xi|^2 \leq a_{ij}^{\alpha\beta}(x, t)\xi_\alpha^i \xi_\beta^j \leq \Lambda|\xi|^2, \quad (1.2)$$

where $\Lambda > 1$, $\xi \in \mathbb{R}^{(q+1)N}$, $(x, t) \in Q_T$.

We say $u \in V_2(Q_T)$ (see Section 2) is a weak solution of (1.1), if for any vector-valued function $\psi \in C_0^\infty(Q_T)$,

$$\iint_{Q_T} [u_t^i \psi + a_{ij}^{\alpha\beta} X_\alpha \psi^i X_\beta u^j] dz = \iint_{Q_T} [g_i \psi + f_i^\alpha X_\alpha \psi] dz.$$

Now, we state the main result of this paper.

Theorem 1.1 *Under the assumption (H), let $u \in V_2(Q_T)$ be a weak solution of (1.1) in Q_T . Suppose that there exists γ , such that $\lambda < \gamma < Q + 2$ and the function $\frac{r^{\gamma-\lambda}}{\varphi^2(r)}$ ($r > 0$) is almost increasing (see Section 2 below). Then $Xu \in L_\varphi^{2,\lambda}(Q')$ for any $Q' \Subset Q'' \Subset Q_T$. Moreover, the following estimate holds*

$$\|Xu\|_{L_\varphi^{2,\lambda}(Q')}^2 \leq c(\|Xu\|_{L^2(Q'')} + \|f\|_{L_\varphi^{2,\lambda}(Q_T)} + \|g\|_{L_\varphi^{2,\lambda}(Q_T)}).$$

The plan of the paper is organized as follows. In Section 2, we introduce some function spaces such as generalized Morrey spaces, generalized Sobolev spaces, and give some known results which will be used. Section 3 is devoted to deducing a Caccioppoli inequality (Lemma 3.1) and L^2 estimates for derivatives (with respect to vector fields and t) of weak solutions of (3.1) (Lemma 3.4). Using the reverse Hölder inequality on the homogeneous space, we prove a higher integrability to (3.1) (see Theorem 3.9). With the help of the results in Sections 2 and 3, we complete the proof of Theorem 1.1 in Section 4.

2 Preliminaries

In this section we introduce some preparatory material related to Hörmander's vector fields and state some function spaces. Several known results which will be used later are collected.

For every multi-index $I = (i_1, i_2, \dots, i_k)$, we denote the length of I by $|I| = k$, and set

$$X_I = X_{i_1} X_{i_2} \dots X_{i_k}, X_\beta = [X_{\beta_d}, [X_{\beta_{d-1}}, \dots [X_{\beta_2}, X_{\beta_1}] \dots]].$$

The length of commutator X_β is denoted by $|\beta| = d$.

Definition 2.1 (*Carnot-Carathéodory distance*). *An absolutely continuous curve $\gamma : [0, T] \rightarrow \Omega$ is called a sub-unit curve with respect to the system X , if $\gamma'(t)$ exists and satisfies that for any $\xi \in \mathbb{R}^n$,*

$$\langle \gamma'(t), \xi \rangle^2 \leq \sum_{j=1}^q \langle X_j(\gamma(t)), \xi \rangle^2, \text{ a.e. } t \in [0, T].$$

The length of γ is denoted by $l_S(\gamma) = T$. Given any $x, y \in \Omega$, we stand for the collection of all sub-unit curves connecting x and y by $\Phi(x, y)$ and define

$$d_X(x, y) = \inf\{l_S(\gamma) : \gamma \in \Phi(x, y)\}.$$

Note that the function $d_X(x, y)$ is finite for any $x, y \in \Omega$, and d_X is really a distance in Ω . One calls that d_X is a Carnot-Carathéodory distance.

A metric ball of center x and radius R is denoted by

$$B_R(x) = B(x, R) = \{y \in \Omega : d_X(x, y) < R\}.$$

When we do not consider the center of a ball, we will simply write B_R instead of $B(x, R)$.

Due to [14], for $\Omega \subset \mathbb{R}^n$, there exist constants $C_D, R_D > 0$ such that for every $x_0 \in \Omega$ and $0 < R < R_D$, one has

$$|B(x_0, 2R)| \leq C_D |B(x_0, R)|.$$

Moreover, for every $R \leq R_D$ and $\tau \in (0, 1)$, we have

$$|B_{\tau R}| \geq C_D^{-1} \tau^Q |B_R|. \quad (2.1)$$

Through out of this paper, we denote $Q_T = \Omega \times (0, T]$ and $z_0 = (x_0, t_0) \in Q_T \subset \mathbb{R}^{n+1}$. A parabolic cylinder with vertex at z_0 is denoted by

$$Q_R(z_0) = B_R(x_0) \times (t_0 - R^2, t_0].$$

In the sequel, let us denote $I_R(t_0) = (t_0 - R^2, t_0]$ and the parabolic boundary of Q_R by $\partial_p Q_R$. Denote the Lebesgue measure of $B(x, R)$ in the n -dimensional space by $|B(x, R)|$, and the Lebesgue measure of $Q_R(z_0)$ in the $n+1$ -dimensional space by $|Q_R(z_0)|$.

Definition 2.2 (*Almost increasing function, see [11]*). A function $h : [0, d_0] \rightarrow [0, \infty)$ is said almost increasing, where $d_0 > 0$, if there exists $K_h \geq 1$, such that for any $0 \leq s \leq t \leq d_0$, the following holds

$$h(s) \leq K_h h(t).$$

Definition 2.3 Let $1 \leq p < +\infty, 0 \leq \lambda < Q+2$ and φ be a continuous function on $[0, d]$ such that $\varphi > 0$ on $(0, d]$, where d is the diameter of Q_T . We say that $f \in L^p(Q_T)$ belongs to a generalized Morrey space $L_{\varphi}^{p, \lambda}(Q_T)$, if

$$\|f\|_{L_{\varphi}^{p, \lambda}} = \sup_{z_0 \in Q_T, 0 \leq \rho \leq d} \frac{1}{\varphi(\rho)} (\rho^{-\lambda} \iint_{Q_T \cap Q_{\rho}(z_0)} |f|^p dz)^{\frac{1}{p}} < \infty.$$

It is easy to prove that the space $L_{\varphi}^{p, \lambda}(Q_T)$ is a Banach space as in [13].

Definition 2.4 (*BMO and VMO space*). For any $f \in L^1(Q_T)$, we set

$$\eta(r) = \sup_{z_0 \in Q, 0 \leq \rho \leq r} \left(\frac{1}{|Q_T \cap Q_{\rho}(z_0)|} \iint_{Q_T \cap Q_{\rho}(z_0)} |f(z) - f_{Q_T \cap Q_{\rho}(z_0)}(z)| dz \right),$$

where $f_{Q_T \cap Q_{\rho}(z_0)} = \frac{1}{|Q_T \cap Q_{\rho}(z_0)|} \iint_{Q_T \cap Q_{\rho}(z_0)} f(z) dz$. If $\sup_{r>0} \eta(r) < \infty$, then it says $f \in BMO(Q_T)$ (Bounded Mean Oscillation). Moreover, if $\eta(r) \rightarrow 0$ as $r \rightarrow 0$, then we call $f \in VMO(Q_T)$ (Vanishing Mean Oscillation).

Definition 2.5 (*Generalized Sobolev space*). The space

$$V_2(Q_T) = \{u : u \in L^{\infty}(0, T; L^2(Q_T)), Xu \in L^2(Q_T)\}$$

is called a generalized Sobolev space, where $|Xu| = \left(\sum_{i=1}^q |X_i u|^2 \right)^{\frac{1}{2}}$.

Lemma 2.6 (*see [11]*). Let H be a non-negative almost increasing function in $[0, R_0]$ and F a positive function on $(0, R_0]$. Suppose that H and F satisfy

(1) There exist positive constants A, B, ε and β such that for any $0 \leq \rho \leq R \leq R_0$,

$$H(\rho) \leq (A(\frac{\rho}{R})^\beta + \varepsilon)H(R) + BF(R); \quad (2.2)$$

(2) There exists $\gamma \in (0, \beta)$ such that $\frac{\rho^\gamma}{F(\rho)}$ is almost increasing in $(0, R_0]$.

Then there exist $\varepsilon_0 = \varepsilon_0(A, \beta, \gamma)$ and $C = C(A, \beta, \gamma, K_H)$ such that if $\varepsilon < \varepsilon_0$, one has

$$H(\rho) \leq C \frac{F(\rho)}{F(R)} H(R) + CBF(\rho). \quad (2.3)$$

The following technical lemma is from [5].

Lemma 2.7 *Let $f(t)$ be a bounded nonnegative function on $[T_0, T_1]$, $T_1 > T_0 \geq 0$. Suppose that for any s and t , $T_0 \leq t < s \leq T_1$, f satisfies*

$$f(t) \leq \theta f(s) + \frac{A}{(s-t)^\alpha} + B,$$

where θ, A, B, α are nonnegative constants and $\theta < 1$. Then for any $T_0 \leq \rho < R \leq T_1$, one has

$$f(\rho) \leq C[\frac{A}{(R-\rho)^\alpha} + B],$$

where C depends only on α .

3 Homogeneous parabolic system with constant coefficients

Let us consider the homogeneous degenerate parabolic system

$$u_t^i + X_\alpha^*(a_{ij}^{\alpha\beta} X_\beta u^j) = 0, \quad (3.1)$$

where coefficients $a_{ij}^{\alpha\beta}$ are constants and satisfy (1.2). We will establish a Caccioppoli inequality and L^2 estimates for derivatives (with respect to vector fields X_1, \dots, X_q and the variable t) of weak solutions of (3.1) by extending results in [20]. Using the reverse Hölder inequality on the homogeneous space, a higher integrability to (3.1) is proved. To simplify the notations, in the sequel, $Q_R(z_0)$, $B_R(x_0)$, $I_R(t_0)$ and $dxdt$ are written as Q_R , B_R , I_R and dz , respectively.

Lemma 3.1 *(Caccioppoli inequality). Let $u \in V_2(Q_T)$ be a weak solution of (3.1). Then for any $Q_R \subset Q_T$ and $\rho < R$,*

$$\sup_{I_\rho} \int_{B_\rho} |u|^2 dx + \iint_{Q_\rho} |Xu|^2 dxdt \leq \frac{c}{(R-\rho)^2} \iint_{Q_R} |u|^2 dxdt. \quad (3.2)$$

Furthermore, for any $b \in \mathbb{R}$, it follows

$$\sup_{I_\rho} \int_{B_\rho} |u - b|^2 dx + \iint_{Q_\rho} |Xu|^2 dxdt \leq \frac{c}{(R-\rho)^2} \iint_{Q_R} |u - b|^2 dxdt \quad (3.3)$$

and

$$\iint_{Q_\rho} |u_t|^2 dx dt \leq \frac{c}{(R-\rho)^4} \iint_{Q_R} |u|^2 dx dt. \quad (3.4)$$

Proof. Given $B_\rho \subset B_R \subset \Omega$, choose a test function $f_i(x) = u^i \xi^2(x) \eta(t)$ with

$$\xi(x) \in C_0^\infty(B_R), 0 \leq \xi \leq 1, |X\xi| \leq \frac{C}{R-\rho}, \xi = 1 \text{ (in } B_\rho)$$

and

$$\eta(t) = \begin{cases} \frac{t-(t_0-R^2)}{R^2-\rho^2}, & t \in (t_0-R^2, t_0-\rho^2), \\ 1, & t \in [t_0-\rho^2, t_0]. \end{cases}$$

Multiplying both sides of (3.1) by $f_i(x)$ and integrating on $Q'_R = B_R(x_0) \times (t_0-R^2, t]$, we get

$$\begin{aligned} 0 &= \iint_{Q'_R} [u_t^i + X_\alpha^*(a_{ij}^{\alpha\beta} X_\beta u^j)] u^i \xi^2 \eta dz \\ &= \iint_{Q'_R} \left[\left(\frac{1}{2} |u|^2 \eta \right)_t \xi^2 - \frac{1}{2} |u|^2 \xi^2 \eta_t + a_{ij}^{\alpha\beta} \xi^2 \eta X_\alpha u^i X_\beta u^j \right. \\ &\quad \left. + 2a_{ij}^{\alpha\beta} u^i \xi \eta X_\alpha \xi X_\beta u^j \right] dz, \end{aligned}$$

and then by moving terms,

$$\begin{aligned} &\iint_{Q'_R} \left[\left(\frac{1}{2} |u|^2 \eta \right)_t \xi^2 + a_{ij}^{\alpha\beta} \xi^2 \eta X_\alpha u^i X_\beta u^j \right] dz \\ &= \iint_{Q'_R} \left[\frac{1}{2} |u|^2 \xi^2 \eta_t - 2a_{ij}^{\alpha\beta} u^i \xi \eta X_\alpha \xi X_\beta u^j \right] dz \\ &\leq \iint_{Q'_R} \frac{1}{2} |u|^2 \xi^2 \eta_t dz + \varepsilon \iint_{Q'_R} \xi^2 \eta |Xu|^2 dz + C_\varepsilon \iint_{Q'_R} \eta |u|^2 |X\xi|^2 dz. \quad (3.5) \end{aligned}$$

Using

$$\begin{aligned} \iint_{Q'_R} \left(\frac{1}{2} |u|^2 \eta \right)_t \xi^2 dz &= \int_{B_R} \int_{(t_0-R^2, t]} \left(\frac{1}{2} |u|^2 \eta \right)_t \xi^2 dt dx \\ &= \eta(t) \int_{B_R} \frac{1}{2} |u|^2 \xi^2 dx, \end{aligned}$$

and (1.2), we have from (3.5) that

$$\begin{aligned} &\eta(t) \int_{B_R} \frac{1}{2} |u|^2 \xi^2 dx + C \iint_{Q'_R} \xi^2 \eta |Xu|^2 dz \\ &\leq \iint_{Q'_R} \frac{1}{2} |u|^2 \xi^2 \eta_t dz + \varepsilon \iint_{Q'_R} \xi^2 \eta |Xu|^2 dz + C_\varepsilon \iint_{Q'_R} \eta |u|^2 |X\xi|^2 dz. \end{aligned}$$

In the light of properties of $\xi(x)$ and $\eta(t)$, it implies

$$\begin{aligned} & \eta(t) \int_{B_R} |u|^2 \xi^2 dx + \iint_{Q'_R} \xi^2 \eta |Xu|^2 dz \\ & \leq C_\varepsilon \iint_{Q'_R} |u|^2 \xi^2 \eta_t dz + C_\varepsilon \iint_{Q'_R} \eta |u|^2 |X\xi|^2 dz \\ & \leq C_\varepsilon \iint_{Q'_R} |u|^2 \left(\frac{1}{R^2 - \rho^2} + \frac{C}{(R - \rho)^2} \right) dz, \end{aligned}$$

thus

$$\sup_{I_\rho} \int_{B_\rho} |u|^2 dx + \iint_{Q_\rho} |Xu|^2 dx dt \leq \frac{C}{(R - \rho)^2} \iint_{Q_R} |u|^2 dx dt,$$

namely, (3.2) is proved.

The proof of (3.3) is similar to that of (3.2), just taking the test function $f_i(x) = (u^i - b)\xi^2(x)\eta(t)$ instead. We omit the details.

Now we come to prove (3.4). Let $\rho \leq s < l \leq R$ with $l - s = s - \rho$ and $f_i(x) = u_t^i \xi_1^2(x)\eta_1(t)$ be a test function with

$$\xi_1(x) \in C_0^\infty(B_s), 0 \leq \xi_1 \leq 1, |X\xi_1| \leq \frac{C}{s - \rho}, \xi_1 = 1 \text{ (in } B_\rho)$$

and

$$\eta_1(t) = \begin{cases} \frac{t - (t_0 - s^2)}{s^2 - \rho^2}, & t \in (t_0 - s^2, t_0 - \rho^2), \\ 1, & t \in [t_0 - \rho^2, t_0]. \end{cases}$$

Multiplying both sides in (3.1) by $f_i(x)$ and integrating on Q_s , one gets

$$0 = \iint_{Q_s} [(u_t^i)^2 \xi_1^2 \eta_1 + a_{ij}^{\alpha\beta} \xi_1^2 \eta_1 X_\alpha u_t^i X_\beta u^j + 2a_{ij}^{\alpha\beta} u_t^i \xi_1 \eta_1 X_\alpha \xi_1 X_\beta u^j] dz,$$

then

$$\begin{aligned} \iint_{Q_s} \xi_1^2 \eta_1 |u_t|^2 dz & \leq C \iint_{Q_s} [\xi_1^2 \eta_1 |Xu_t| |Xu| + \eta_1 |\xi_1 u_t| |Xu| |X\xi_1|] dz \\ & \leq \varepsilon \iint_{Q_s} \xi_1^2 \eta_1 |Xu_t|^2 dz + \frac{C}{\varepsilon} \iint_{Q_s} \xi_1^2 \eta_1 |Xu|^2 dz \\ & \quad + \frac{1}{2} \iint_{Q_s} \eta_1 |\xi_1 u_t|^2 dz + C \iint_{Q_s} \eta_1 |Xu|^2 |X\xi_1|^2 dz. \end{aligned}$$

Noting properties of $\xi_1(x)$ and $\eta_1(t)$, it yields

$$\begin{aligned} \iint_{Q_\rho} |u_t|^2 dz & \leq 2\varepsilon \iint_{Q_s} |Xu_t|^2 dz \\ & \quad + \frac{C}{\varepsilon} \iint_{Q_s} |Xu|^2 dz + \frac{C}{(s - \rho)^2} \iint_{Q_s} |Xu|^2 dz. \end{aligned} \quad (3.6)$$

Since u_t is still a weak solution of (3.1), we apply (3.2) to u_t and have

$$\iint_{Q_s} |Xu_t|^2 dz \leq \frac{C}{(l-s)^2} \iint_{Q_l} |u_t|^2 dz$$

and

$$\iint_{Q_s} |Xu|^2 dz \leq \frac{C}{(l-s)^2} \iint_{Q_l} |u|^2 dz.$$

Inserting the above two inequalities into (3.6) and using $l-s = s-\rho$, it obtains

$$\begin{aligned} & \iint_{Q_\rho} |u_t|^2 dz \\ & \leq \frac{2\varepsilon C}{(l-s)^2} \iint_{Q_l} |u_t|^2 dz + \frac{C}{\varepsilon(l-s)^2} \iint_{Q_l} |u|^2 dz + \frac{C}{(s-\rho)^2(l-s)^2} \iint_{Q_l} |u|^2 dz \\ & \leq \frac{\varepsilon C}{(l-s)^2} \iint_{Q_l} |u_t|^2 dz + \frac{C}{\varepsilon(l-s)^2} \iint_{Q_l} |u|^2 dz + \frac{C}{(l-s)^4} \iint_{Q_l} |u|^2 dz. \end{aligned}$$

Taking $\varepsilon = \frac{(l-s)^2}{4C}$, it follows

$$\iint_{Q_\rho} |u_t|^2 dz \leq \frac{1}{4} \iint_{Q_l} |u_t|^2 dz + \frac{C}{(l-\rho)^4} \iint_{Q_l} |u|^2 dz$$

and then (3.4) from Lemma 2.7. ■

Remark 3.2 *Checking carefully the proof of Lemma 3.1, one find that conclusions in Lemma 3.1 are still hold for the homogeneous parabolic system with variable coefficients, provided coefficients are bounded and satisfy (1.2). It will be used in Section 4.*

Lemma 3.3 *Let $u \in C^\infty(Q_T)$, $B_R \subset \Omega$ and $I_R \subset (0, T)$. Then*

(i) when $k > \frac{Q}{2}$, there exist positive constants R_0 and c such that for any $R \leq R_0$,

$$\sup_{x \in B_{R/4}} |u(x, t)| \leq c |B_R|^{-\frac{1}{2}} \sum_{|I| \leq k} R^{|I|} \|X_I u(x, t)\|_{L^2(B_R)}. \quad (3.7)$$

(ii) when $k > 1$, there exist positive constants R_0 and c such that for any $R \leq R_0$,

$$\sup_{t \in I_{R/4}} |u(x, t)| \leq c \sum_{2m \leq k} R^{2m-1} \|\partial_t^m u(x, t)\|_{L^2(I_R)}. \quad (3.8)$$

The first statement is from Proposition 2.4 in [20]. The second is easily proved by the same way in [20]. We omit it here.

Lemma 3.4 *Let $u \in V_2(Q_T)$ be a weak solution of (3.1). Then $u \in C^\infty(Q_T)$ and for any positive integer k , it follows*

$$\sum_{|I| \leq k} \iint_{Q_{R/2^k}} |X_I u|^2 dz \leq \frac{c}{R^{2k}} \iint_{Q_R} |u|^2 dz \quad (3.9)$$

and

$$\sum_{|I|+2m \leq k} \iint_{Q_{R/2^k}} |X_I \partial_t^m u|^2 dz \leq \frac{c}{R^{2k}} \iint_{Q_R} |u|^2 dz. \quad (3.10)$$

Proof. Denote $M^k(\Omega) = \{u \in L^2(\Omega), X_I u \in L^2(\Omega), |I| \leq k\}$ and $Lu = u_t^i + X_\alpha^* (a_{ij}^{\alpha\beta} X_\beta u^j)$. Since u is a weak solution of (3.1) and L is hypoelliptic, we deduce that u belongs to $C^\infty(Q_T)$ from $Lu = 0$.

Let us test (3.9) by the induction on k . When $k = 1$, setting $\rho = \frac{R}{2}$ in (3.2) leads to

$$\iint_{Q_{R/2}} |X_I u|^2 dz \leq \frac{c}{R^2} \iint_{Q_R} |u|^2 dz.$$

Assuming that (3.9) is true if $|I| \leq k-1$ ($k \geq 2$), we show that (3.9) is still true when $|I| = k$.

Let $\xi(x)\eta(t)$ be a cutoff function with

$$\xi(x) \in C_0^\infty(B_{R/2^{k-1}}), 0 \leq \xi \leq 1, |X_I \xi| \leq \frac{C}{R^{|I|}}, \xi = 1 \text{ (in } B_{R/2^k})$$

and

$$\eta(t) = \begin{cases} \frac{t - (t_0 - (R/2^{k-1})^2)}{(R/2^{k-1})^2 - (R/2^k)^2}, & t \in (t_0 - (R/2^{k-1})^2, t_0 - (R/2^k)^2), \\ 1, & t \in [t_0 - (R/2^k)^2, t_0]. \end{cases}$$

Denote $\tilde{L}u = a_{ij}^{\alpha\beta} X_\alpha X_\beta u^j$. Recalling $Lu = 0$ and $X_\alpha^* = -X_\alpha + c_\alpha$, one sees

$$\tilde{L}u = a_{ij}^{\alpha\beta} c_\alpha X_\beta u^j + u_t^i.$$

Due to regularity result by Rothschild and Stein ([16]) to the operator \tilde{L} , we have

$$\begin{aligned}
& \sum_{|I|=k} \iint_{Q_{R/2^k}} |X_I u|^2 dz \\
& \leq \sum_{|I| \leq k} \int_{I_{R/2^k}} \|X_I u\|_{L^2(B_{R/2^k})}^2 dt \leq \int_{I_{R/2^{k-1}}} \|\xi \eta u\|_{M^k(B_{R/2^{k-1}})}^2 dt \\
& \leq c \int_{I_{R/2^{k-1}}} [\|\tilde{L}(\xi \eta u)\|_{M^{k-2}(B_{R/2^{k-1}})}^2 + \|\xi \eta u\|_{L^2(B_{R/2^{k-1}})}^2] dt \\
& \leq c \int_{I_{R/2^{k-1}}} [\|a_{ij}^{\alpha\beta} c_\alpha X_\beta (\xi \eta u)^j + (\xi \eta u)_t^i\|_{M^{k-2}(B_{R/2^{k-1}})}^2 + \|\xi \eta u\|_{L^2(B_{R/2^{k-1}})}^2] dt \\
& \leq c \int_{I_{R/2^{k-1}}} [\|a_{ij}^{\alpha\beta} c_\alpha X_\beta (\xi \eta u)^j\|_{M^{k-2}(B_{R/2^{k-1}})}^2 + \\
& \quad + \|(\xi \eta u)_t^i\|_{M^{k-2}(B_{R/2^{k-1}})}^2 + \|\xi \eta u\|_{L^2(B_{R/2^{k-1}})}^2] dt. \tag{3.11}
\end{aligned}$$

Let us denote

$$\begin{aligned}
I & \equiv c \int_{I_{R/2^{k-1}}} \|a_{ij}^{\alpha\beta} c_\alpha X_\beta (\xi \eta u)^j\|_{M^{k-2}(B_{R/2^{k-1}})}^2 dt, \\
II & \equiv c \int_{I_{R/2^{k-1}}} \|(\xi \eta u)_t^i\|_{M^{k-2}(B_{R/2^{k-1}})}^2 dt.
\end{aligned}$$

We first estimate II . By properties of $\xi(x)$ and $\eta(t)$,

$$\begin{aligned}
II & = c \int_{I_{R/2^{k-1}}} \|(\xi \eta u)_t\|_{M^{k-2}(B_{R/2^{k-1}})}^2 dt \\
& = c \sum_{|I| \leq k-2} \iint_{Q_{R/2^{k-1}}} |X_I (\xi \eta_t u + \xi \eta u_t)|^2 dz \\
& \leq c \sum_{|I| \leq k-2} \iint_{Q_{R/2^{k-1}}} [|\eta_t u X_I \xi|^2 + |\xi \eta_t X_I u|^2 + |\eta u_t X_I \xi|^2 + |\xi \eta X_I u_t|^2] dz, \\
& \leq c \sum_{|I| \leq k-2} \frac{1}{R^{2(|I|+2)}} \iint_{Q_{R/2^{k-1}}} |u|^2 dz + c \sum_{|I| \leq k-2} \frac{1}{R^4} \iint_{Q_{R/2^{k-1}}} |X_I u|^2 dz \\
& \quad + c \sum_{|I| \leq k-2} \frac{1}{R^{2|I|}} \iint_{Q_{R/2^{k-1}}} |u_t|^2 dz + c \sum_{|I| \leq k-2} \iint_{Q_{R/2^{k-1}}} |X_I u_t|^2 dz. \tag{3.12}
\end{aligned}$$

From the assertion for $|I| \leq k-1$, it follows

$$\sum_{|I| \leq k-2} \iint_{Q_{R/2^{k-1}}} |X_I u|^2 dz \leq \frac{c}{R^{2(k-2)}} \iint_{Q_R} |u|^2 dz. \tag{3.13}$$

On the other hand, since u_t is still a weak solution of (3.1), we apply (3.4) to u_t and derive

$$\sum_{|I| \leq k-2} \iint_{Q_{R/2^{k-1}}} |X_I u_t|^2 dz \leq \frac{c}{R^{2(k-2)}} \iint_{Q_{R/2}} |u_t|^2 dz \leq \frac{c}{R^{2k}} \iint_{Q_R} |u|^2 dz. \quad (3.14)$$

Inserting (3.13) and (3.14) into (3.12), it shows from (3.4) that

$$II \leq \frac{c}{R^{2k}} \iint_{Q_{R/2^{k-1}}} |u|^2 dz + \frac{c}{R^{2k}} \iint_{Q_R} |u|^2 dz \leq \frac{c}{R^{2k}} \iint_{Q_R} |u|^2 dz. \quad (3.15)$$

Now let us handle I . Since $c_\alpha \in C_0^\infty$, $|X_I c_\alpha|$ is bounded. By properties of $\xi(x)$ and $\eta(t)$,

$$\begin{aligned} I &= c \int_{I_{R/2^{k-1}}} \left(\sum_{|I| \leq k-2} \int_{B_{R/2^{k-1}}} \left| a_{ij}^{\alpha\beta} \eta X_I (c_\alpha X_\beta (\xi u)^j) \right|^2 dx \right) dt \\ &\leq c \sum_{|I| \leq k-2} \iint_{Q_{R/2^{k-1}}} [|\xi X_\beta u^j X_I c_\alpha|^2 + |c_\alpha X_\beta u^j X_I \xi|^2 + |c_\alpha \xi X_I X_\beta u^j|^2] dz \\ &\quad + c \sum_{|I| \leq k-2} \iint_{Q_{R/2^{k-1}}} [|u^j X_\beta \xi X_I c_\alpha|^2 + |c_\alpha X_\beta \xi X_I u^j|^2 + |c_\alpha u^j X_I X_\beta \xi|^2] dz \\ &\leq c \sum_{|I| \leq k-2} \iint_{Q_{R/2^{k-1}}} [|X_\beta u^j|^2 + \frac{c}{R^{2|I|}} |X_\beta u^j|^2 + |X_I X_\beta u^j|^2] dz \\ &\quad + c \sum_{|I| \leq k-2} \iint_{Q_{R/2^{k-1}}} [\frac{1}{R^2} |u^j|^2 + \frac{1}{R^2} |X_I u^j|^2 + \frac{c}{R^{2(|I|+1)}} |u^j|^2] dz. \end{aligned} \quad (3.16)$$

By the assertion for $|I| \leq k-1$,

$$\sum_{|I| \leq k-2} \iint_{Q_{R/2^{k-1}}} |X_I X_\beta u^j|^2 dz \leq \frac{c}{R^{2(k-1)}} \iint_{Q_R} |u|^2 dz \leq \frac{c}{R^{2k}} \iint_{Q_R} |u|^2 dz$$

and

$$\sum_{|I| \leq k-2} \iint_{Q_{R/2^{k-1}}} |X_I u^j|^2 dz \leq \frac{c}{R^{2(k-2)}} \iint_{Q_{R/2}} |u|^2 dz \leq \frac{c}{R^{2(k-2)}} \iint_{Q_R} |u|^2 dz.$$

Inserting the above two inequalities into (3.16) yields

$$\begin{aligned} I &\leq \frac{c}{R^2} \iint_{Q_{R/2^{k-2}}} |u|^2 dz + \frac{c}{R^{2(k-1)}} \iint_{Q_R} |u|^2 dz + \frac{c}{R^{2k}} \iint_{Q_R} |u|^2 dz \\ &\leq \frac{c}{R^{2k}} \iint_{Q_R} |u|^2 dz. \end{aligned} \quad (3.17)$$

Putting (3.15) and (3.17) into (3.11), we get

$$\sum_{|I|=k} \iint_{Q_{R/2^k}} |X_I u|^2 dz \leq \frac{c}{R^{2k}} \iint_{Q_R} |u|^2 dz,$$

hence (3.9) is proved.

The proof of (3.10) is easy. In fact, since $\partial_t^m u$ is also a weak solution of (3.1), it shows by applying (3.9) to $\partial_t^m u$ and noting (3.4) that

$$\begin{aligned}
& \sum_{|I|+2m \leq k} \iint_{Q_{R/2^k}} |X_I \partial_t^m u|^2 dx dt \\
& \leq \sum_{|I|+2m \leq k} \frac{c}{R^{2|I|}} \iint_{Q_{R/2^{2m}}} |\partial_t^m u|^2 dx dt \\
& \leq \sum_{|I|+2m \leq k} \frac{c}{R^{2|I|}} \frac{c}{R^{4m}} \iint_{Q_R} |u|^2 dx dt \\
& \leq \frac{c}{R^{2k}} \iint_{Q_R} |u|^2 dx dt.
\end{aligned}$$

■

Lemma 3.5 (Sobolev-Poincaré inequality, see [7], [12] and references therein). *For any open set Ω' , $\bar{\Omega}' \Subset \Omega$, there exist positive constants R_0 and c such that for any $x_0 \in \Omega'$, $0 < R \leq R_0$, $u \in C^\infty(\bar{B}_R)$,*

$$\left(\frac{1}{|B_R|} \int_{B_R} |u - u_R|^{q'} dx \right)^{\frac{1}{q'}} \leq cR \left(\frac{1}{|B_R|} \int_{B_R} \sum_{i=1}^q |X_i u|^{p'} dx \right)^{\frac{1}{p'}},$$

where $1 < p' < Q$, $1 \leq q' < \frac{p'Q}{Q-p'}$, $u_R(t) = \frac{1}{|B_R|} \int_{B_R} u(x, t) dx$, R_0 and c depend on Ω' and Ω .

If $u \in C_0^\infty(B_R)$, then for all $1 \leq q' \leq \frac{p'Q}{Q-p'}$,

$$\left(\frac{1}{|B_R|} \int_{B_R} |u|^{q'} dx \right)^{\frac{1}{q'}} \leq cR \left(\frac{1}{|B_R|} \int_{B_R} \sum_{i=1}^q |X_i u|^{p'} dx \right)^{\frac{1}{p'}}. \quad (3.18)$$

In particular, if $p' = q' = 2$, then

$$\int_{B_R} |u|^2 dx \leq cR^2 \int_{B_R} \sum_{i=1}^q |X_i u|^2 dx; \quad (3.19)$$

if $p' = 2$, $q' = \frac{2Q}{Q-2}$, then

$$\left(\int_{B_R} |u|^{\frac{2Q}{Q-2}} dx \right)^{\frac{Q-2}{2Q}} \leq c \left(\int_{B_R} |Xu|^2 dx \right)^{\frac{1}{2}}. \quad (3.20)$$

Lemma 3.6 *Let $u \in V_2(Q_T)$ be a weak solution of (3.1) in Q_T and $Q_R \subset Q_T$. Then for any $0 \leq \rho \leq R$,*

$$\iint_{Q_\rho} |u|^2 dz \leq C \left(\frac{\rho}{R} \right)^{Q+2} \iint_{Q_R} |u|^2 dz.$$

Proof. Let k_1 and k_2 be fixed integers such that $k_1 > \frac{Q}{2}$ and $k_2 > 1$. If $\rho \geq \frac{R}{2^{k_1+k_2+2}}$, then the conclusion is obvious. If $\rho < \frac{R}{2^{k_1+k_2+2}}$, then by (2.1), (3.7) and (3.8),

$$\begin{aligned}
\iint_{Q_\rho} |u|^2 dxdt &\leq \int_{I_\rho} |B_\rho| \sup_{B_{R/2^{k_1+k_2+2}}} |u|^2 dt \\
&\leq c |B_\rho| \int_{I_\rho} \left(\sum_{|I| \leq k_1} |B_{R/2^{k_1+k_2}}|^{-1} R^{2|I|} \int_{B_{R/2^{k_1+k_2}}} |X_I u|^2 dx \right) dt \\
&\leq c \frac{|B_\rho|}{|B_R|} \sum_{|I| \leq k_1} R^{2|I|} \int_{B_{R/2^{k_1+k_2}}} |I_\rho| \sup_{I_\rho} |X_I u|^2 dx \\
&\leq c \rho^2 \frac{|B_\rho|}{|B_R|} \sum_{|I| \leq k_1} R^{2|I|} \int_{B_{R/2^{k_1+k_2}}} \sum_{2m \leq k_2} R^{4m-2} \int_{I_{R/2^{k_1+k_2}}} |\partial_t^m X_I u|^2 dt dx \\
&\leq c \left(\frac{\rho}{R} \right)^2 \frac{|B_\rho|}{|B_R|} \sum_{|I|+2m \leq k_1+k_2} R^{2(|I|+2m)} \iint_{Q_{R/2^{k_1+k_2}}} |\partial_t^m X_I u|^2 dz.
\end{aligned}$$

Applying (3.10) leads to

$$\iint_{Q_\rho} |u|^2 dxdt \leq c \left(\frac{\rho}{R} \right)^2 \frac{|B_\rho|}{|B_R|} \iint_{Q_R} |u|^2 dz \leq c \left(\frac{\rho}{R} \right)^{Q+2} \iint_{Q_R} |u|^2 dz,$$

where we have used the definition of Q and the fact that $|B_R|$ can be approximated by some polynomial in R , see [7],[14]. ■

Lemma 3.7 Suppose that $u \in V_2(Q_T)$ is a weak solution of (3.1), $Q_R(z_0) \subset Q_T$ and $u = 0$ on $\partial_p Q_R$. Then for any $0 \leq \rho \leq R$, it follows

$$\iint_{Q_\rho} |Xu|^2 dxdt \leq c \left(\frac{\rho}{R} \right)^{Q+2} \iint_{Q_R} |Xu|^2 dxdt.$$

Proof. Let k_1 and k_2 be fixed integers such that $k_1 > \frac{Q}{2}$ and $k_2 > 1$. If $\rho \geq \frac{R}{2^{k_1+k_2+3}}$, then the conclusion holds; if $\rho < \frac{R}{2^{k_1+k_2+3}}$, then by (3.7) and (3.8),

$$\begin{aligned}
\iint_{Q_\rho} |X_i u|^2 dxdt &\leq \int_{I_\rho} |B_\rho| \sup_{B_{R/2^{k_1+k_2+3}}} |X_i u|^2 dt \\
&\leq c |B_\rho| \int_{I_\rho} \left(\sum_{|I| \leq k_1} |B_{R/2^{k_1+k_2+1}}|^{-1} R^{2|I|} \int_{B_{R/2^{k_1+k_2+1}}} |X_I X_i u|^2 dx \right) dt \\
&\leq c \frac{|B_\rho|}{|B_R|} \sum_{|I| \leq k_1} R^{2|I|} \int_{B_{R/2^{k_1+k_2+1}}} |I_\rho| \sup_{I_\rho} |X_I X_i u|^2 dx \\
&\leq c \rho^2 \frac{|B_\rho|}{|B_R|} \sum_{|I| \leq k_1} R^{2|I|} \int_{B_{R/2^{k_1+k_2+1}}} \sum_{2m \leq k_2} R^{4m-2} \int_{I_{R/2^{k_1+k_2+1}}} |\partial_t^m X_I X_i u|^2 dt dx \\
&\leq c \left(\frac{\rho}{R} \right)^2 \frac{|B_\rho|}{|B_R|} \sum_{|I|+2m \leq k_1+k_2} R^{2(|I|+2m)} \iint_{Q_{R/2^{k_1+k_2+1}}} |\partial_t^m X_I X_i u|^2 dz.
\end{aligned}$$

In virtue of (3.10) and (3.18),

$$\begin{aligned}
\iint_{Q_\rho} |X_i u|^2 dx dt &\leq c \left(\frac{\rho}{R} \right)^2 \frac{|B_\rho|}{|B_R|} \sum_{|I|+2m \leq k} R^{2(|I|+2m)} R^{-2(|I|+2m+1)} \iint_{Q_R} |u|^2 dz \\
&\leq c \left(\frac{\rho}{R} \right)^2 \frac{|B_\rho|}{|B_R|} \frac{1}{R^2} \iint_{Q_R} |u|^2 dz, \\
&\leq c \left(\frac{\rho}{R} \right)^2 \frac{|B_\rho|}{|B_R|} \iint_{Q_R} |Xu|^2 dz.
\end{aligned}$$

Similarly to Lemma 3.6, we end the proof. \blacksquare

We need to define a parabolic distance d_p corresponding to d_X . For $(x, t), (y, s) \in Q_T$, set

$$d_p((x, t), (y, s)) = (d_X(x, y)^2 + |t - s|)^{\frac{1}{2}}.$$

Denote a ball with respect to the distance d_p by

$$B_p((x_0, t_0), R) = \{(x, t) \in Q_T : d_p((x_0, t_0), (x, t)) < R\}.$$

An important fact is that $B_p((x_0, t_0), R)$ is a homogeneous space (see [8], [1, Proposition 3.8]). According to it and

$$Q_R(z) \subset B_p(z, 2R) \subset Q_{2R}(z),$$

we immediately know that the reverse Hölder inequality in [8] (or [21]) is true for parabolic cylinders.

Lemma 3.8 *Let $g \geq 0$ on Q_T . If for some $\hat{q} > 1$ such that for any $Q_{4R} \subset Q_T$,*

$$\frac{1}{|Q_R|} \iint_{Q_R} g^{\hat{q}} dx dt \leq b \left(\frac{1}{|Q_{4R}|} \iint_{Q_{4R}} g dx dt \right)^{\hat{q}} + \theta \iint_{Q_{4R}} g^{\hat{q}} dx dt. \quad (3.21)$$

Then there exist positive constants $b > 1$ and $\theta_0 = \theta_0(\hat{q}, Q_T)$ such that if $\theta < \theta_0$, then $g \in L_{loc}^p(Q_T)$ for any $p \in (\hat{q}, \hat{q} + \varepsilon]$. Moreover, it holds

$$\left(\frac{1}{|Q_R|} \iint_{Q_R} g^p dx dt \right)^{\frac{1}{p}} \leq c \left(\frac{1}{|Q_{4R}|} \iint_{Q_{4R}} g^{\hat{q}} dx dt \right)^{\frac{1}{\hat{q}}},$$

where the positive constants c and ε depend only on b, \hat{q}, θ and Q .

Theorem 3.9 *Let $Q_R \subset Q_{4R} \subset Q_T$ and $u \in V_2(Q_T)$ be a weak solution of (3.1) in Q_T and $u = 0$ on $\partial_p Q_{4R}$. Then there exists a constant $s > 2$ such that $Xu \in L_{loc}^s(Q_T)$. Moreover, the following inequality holds*

$$\left(\frac{1}{|Q_R|} \iint_{Q_R} |Xu|^s dz \right)^{1/s} \leq C \left(\frac{1}{|Q_{4R}|} \iint_{Q_{4R}} |Xu|^2 dz \right)^{1/2}.$$

Proof. Set $2^* = \frac{2Q}{Q-2}$ and $\tilde{q} = \frac{2Q}{Q+2}$. Note

$$\begin{aligned} & \iint_{Q_{2R}} |u(t)|^2 dz \\ & \leq \sup_{I_{2R}} \left(\int_{B_{2R}} |u(t)|^2 dx \right)^{\frac{1}{2}} \cdot \left(\int_{I_{2R}} \left(\int_{B_{2R}} |u(t)|^2 dx \right)^{\frac{1}{2}} dt \right), \end{aligned} \quad (3.22)$$

and denote

$$A \equiv \sup_{I_{2R}} \left(\int_{B_{2R}} |u(t)|^2 dx \right)^{\frac{1}{2}}, B \equiv \int_{I_{2R}} \left(\int_{B_{2R}} |u(t)|^2 dx \right)^{\frac{1}{2}} dt.$$

Now we estimate A and B , respectively. By (3.2) and (3.19),

$$A \leq \frac{c}{R} \left(\iint_{Q_{4R}} |u|^2 dz \right)^{\frac{1}{2}} \leq c \left(\iint_{Q_{4R}} |Xu|^2 dz \right)^{\frac{1}{2}}. \quad (3.23)$$

To B , we have by (3.18) and (3.20) that

$$\begin{aligned} B & \leq \int_{I_{2R}} \left(\int_{B_{2R}} |u(t)|^{\tilde{q}} dx \right)^{\frac{1}{2\tilde{q}}} \left(\int_{B_{2R}} |u(t)|^{2^*} dx \right)^{\frac{1}{2 \cdot 2^*}} dt \\ & \leq c \int_{I_{2R}} \left(R^{\tilde{q}} \int_{B_{2R}} |Xu|^{\tilde{q}} dx \right)^{\frac{1}{2\tilde{q}}} \left(\int_{B_{2R}} |u|^{2^*} dx \right)^{\frac{1}{2 \cdot 2^*}} dt \\ & \leq cR^{\frac{1}{2}} \int_{I_{2R}} \left(\int_{B_{2R}} |Xu|^{\tilde{q}} dx \right)^{\frac{1}{2\tilde{q}}} \left(\int_{B_{2R}} |Xu|^2 dx \right)^{\frac{1}{4}} dt \\ & \leq cR^{\frac{1}{2}} \left(\iint_{Q_{2R}} |Xu|^{\tilde{q}} dz \right)^{\frac{1}{2\tilde{q}}} \left(\int_{I_{2R}} \left(\int_{B_{2R}} |Xu|^2 dx \right)^{\frac{1}{2} \frac{\tilde{q}}{2\tilde{q}-1}} dt \right)^{\frac{2\tilde{q}-1}{2\tilde{q}}} \\ & \leq cR^{\frac{3}{2}-\frac{1}{\tilde{Q}}} \left(\iint_{Q_{4R}} |Xu|^{\tilde{q}} dz \right)^{\frac{1}{2\tilde{q}}} \left(\iint_{Q_{4R}} |Xu|^2 dz \right)^{\frac{1}{4}}. \end{aligned} \quad (3.24)$$

Inserting (3.23) and (3.24) into (3.22) and using Young's inequality,

$$\begin{aligned} & \iint_{Q_{2R}} |u(t)|^2 dz \\ & \leq cR^{\frac{3}{2}-\frac{1}{\tilde{Q}}} \left(\iint_{Q_{4R}} |Xu|^2 dz \right)^{\frac{3}{4}} \cdot \left(\iint_{Q_{4R}} |Xu|^{\tilde{q}} dz \right)^{\frac{1}{2\tilde{q}}} \\ & \leq \varepsilon R^2 \iint_{Q_{4R}} |Xu|^2 dz + C(\varepsilon) R^{-\frac{4}{\tilde{Q}}} \left(\iint_{Q_{4R}} |Xu|^{\tilde{q}} dz \right)^{\frac{2}{\tilde{q}}}. \end{aligned}$$

Returning to (3.2) and using the above inequality lead to

$$\begin{aligned}
& \frac{1}{|Q_R|} \iint_{Q_R} |Xu|^2 dz \\
& \leq \frac{c}{R^2} \frac{1}{|Q_R|} \iint_{Q_{2R}} |u(t)|^2 dz \\
& \leq \varepsilon \frac{1}{|Q_{4R}|} \iint_{Q_{4R}} |Xu|^2 dz + C(\varepsilon) \frac{|Q_{4R}|^{2/\tilde{q}}}{|Q_R|} R^{-\frac{4}{Q}-2} \left(\frac{1}{|Q_{4R}|} \iint_{Q_{4R}} |Xu|^{\tilde{q}} dz \right)^{\frac{2}{\tilde{q}}} \\
& \leq \varepsilon \frac{1}{|Q_{4R}|} \iint_{Q_{4R}} |Xu|^2 dz + C(\varepsilon) \left(\frac{1}{|Q_{4R}|} \iint_{Q_{4R}} |Xu|^{\tilde{q}} dz \right)^{\frac{2}{\tilde{q}}}.
\end{aligned}$$

Let $g = |Xu|^{\tilde{q}}$, $\hat{q} = \frac{2}{\tilde{q}} = \frac{Q+2}{Q} > 1$, $\theta = \varepsilon$. The previous inequality is of the form

$$\frac{1}{|Q_R|} \iint_{Q_R} g^{\hat{q}} dz \leq \theta \frac{1}{|Q_{4R}|} \iint_{Q_{4R}} g^{\hat{q}} dz + C(\varepsilon) \left(\frac{1}{|Q_{4R}|} \iint_{Q_{4R}} g dz \right)^{\hat{q}}.$$

Due to Lemma 3.8, there exists $\varepsilon > 0$ such that for any $p \in [\hat{q}, \hat{q} + \varepsilon)$,

$$\left(\frac{1}{|Q_R|} \iint_{Q_R} |Xu|^{p\tilde{q}} dz \right)^{\frac{1}{p}} \leq c \left(\frac{1}{|Q_{4R}|} \iint_{Q_{4R}} |Xu|^2 dz \right)^{\frac{\tilde{q}}{2}},$$

Denoting $s = p\tilde{q} \in [2, 2 + \varepsilon)$, the proof is finished. ■

Remark 3.10 *It is not hard to find that the conclusion of Theorem 3.9 is still true for the homogeneous parabolic system with variable coefficients, when we check carefully the above proof. It will be useful in Section 4.*

4 Proof of Theorem 1.1

In this section we will prove Theorem 1.1. First step is to establish the following.

Theorem 4.1 *Let $u \in V_2(Q_T)$ be a weak solution of*

$$u_t^i + X_\alpha^*(a_{ij}^{\alpha\beta}(z)X_\beta u^j) = 0,$$

in Q_T . Suppose coefficients $a_{ij}^{\alpha\beta}(z) \in VMO(Q_T)$ and satisfy (1.2). Then for any $0 < \mu < Q + 2$, there exist positive constants R_0 and c such that for any $\rho \leq R \leq \frac{1}{2} \min(R_0, \text{dist}(z_0, \partial_p Q_T))$, it holds

$$\iint_{Q_\rho} |Xu|^2 dxdt \leq c \left(\frac{\rho}{R} \right)^\mu \iint_{Q_R} |Xu|^2 dxdt,$$

where R_0 and c depend on Q , μ , Λ and the VMO modulus of $a_{ij}^{\alpha\beta}$.

Proof. Let w be a weak solution of the following system

$$\begin{cases} w_t^i + X_\alpha^*((a_{ij}^{\alpha\beta})_{z_0,R})X_\beta w^j = 0, & \text{in } Q_R, \\ w = u, & \text{on } \partial_p Q_R, \end{cases} \quad (4.1)$$

where z_0 is a fixed point in Q_R , $(a_{ij}^{\alpha\beta})_{z_0,R} = \frac{1}{|Q_T \cap Q_R|} \iint_{Q_T \cap Q_R} a_{ij}^{\alpha\beta}(z) dz$. Then $v = u - w$ satisfies

$$\begin{cases} v_t^i + X_\alpha^*((a_{ij}^{\alpha\beta})_{z_0,R})X_\beta v^j = X_\alpha^*((a_{ij}^{\alpha\beta})_{z_0,R} - a_{ij}^{\alpha\beta}(z))X_\beta u^j, & \text{in } Q_R, \\ v = 0, & \text{on } \partial_p Q_R. \end{cases} \quad (4.2)$$

Multiplying both sides of (4.2) by v^i and integrating by parts on Q_R ,

$$\begin{aligned} & \iint_{Q_R} (v_t^i v^i + (a_{ij}^{\alpha\beta})_{z_0,R} X_\beta v^j X_\alpha v^i) dz \\ &= \iint_{Q_R} (((a_{ij}^{\alpha\beta})_{z_0,R} - a_{ij}^{\alpha\beta}(z)) X_\beta u^j X_\alpha v^i) dz \\ &\leq \iint_{Q_R} |a_{ij}^{\alpha\beta}(z) - (a_{ij}^{\alpha\beta})_{z_0,R}| |Xu| |Xv| dz \\ &\leq C_\varepsilon \iint_{Q_R} |a_{ij}^{\alpha\beta}(z) - (a_{ij}^{\alpha\beta})_{z_0,R}|^2 |Xu|^2 dz + \varepsilon \iint_{Q_R} |Xv|^2 dz. \end{aligned}$$

Noting $\iint_{Q_R} v_t^i v^i dz = \int_{B_R} dx \int_{t_0-R^2}^{t_0} v^i dv^i \geq 0$ and (1.2), it follows

$$\iint_{Q_R} |Xv|^2 dz \leq C_\varepsilon \iint_{Q_R} |a_{ij}^{\alpha\beta}(z) - (a_{ij}^{\alpha\beta})_{z_0,R}|^2 |Xu|^2 dz. \quad (4.3)$$

From $a_{ij}^{\alpha\beta} \in VMO$, we see that for any $\varepsilon > 0$, there exists $R_0 > 0$ such that for any $R \leq R_0$,

$$\left(\frac{1}{|Q_R|} \iint_{Q_R} |a_{ij}^{\alpha\beta}(z) - (a_{ij}^{\alpha\beta})_{z_0,R}|^{\frac{2s}{s-2}} dz \right)^{\frac{s-2}{s}} < \varepsilon$$

and

$$\begin{aligned} & \iint_{Q_R} |a_{ij}^{\alpha\beta}(z) - (a_{ij}^{\alpha\beta})_{z_0,R}|^2 |Xu|^2 dz \\ &\leq |Q_R| \left(\frac{1}{|Q_R|} \iint_{Q_R} |a_{ij}^{\alpha\beta}(z) - (a_{ij}^{\alpha\beta})_{z_0,R}|^{\frac{2s}{s-2}} dz \right)^{\frac{s-2}{s}} \left(\frac{1}{|Q_R|} \iint_{Q_R} |Xu|^s dz \right)^{2/s} \\ &\leq \varepsilon |Q_R| \left(\frac{1}{|Q_R|} \iint_{Q_R} |Xu|^s dz \right)^{2/s} \\ &\leq \varepsilon \iint_{Q_{4R}} |Xu|^2 dz, \end{aligned}$$

where we have used Theorem 3.9, Remarks 3.2 and 3.10.

Inserting the above inequality into (4.3), we immediately get

$$\iint_{Q_R} |Xv|^2 dz \leq \varepsilon \iint_{Q_{4R}} |Xu|^2 dz.$$

Applying Lemma 3.7 to w ,

$$\begin{aligned} \iint_{Q_\rho} |Xu|^2 dz &\leq 2 \iint_{Q_\rho} |Xv|^2 dz + 2 \iint_{Q_\rho} |Xw|^2 dz \\ &\leq c \iint_{Q_\rho} |Xv|^2 dz + c \left(\frac{\rho}{R}\right)^{Q+2} \iint_{Q_R} |Xw|^2 dz \\ &\leq c \iint_{Q_\rho} |Xv|^2 dz + c \left(\frac{\rho}{R}\right)^{Q+2} \iint_{Q_R} |Xu|^2 dz \\ &\leq c \left(\left(\frac{\rho}{R}\right)^{Q+2} + \varepsilon\right) \iint_{Q_R} |Xu|^2 dz. \end{aligned}$$

The proof is reached by using Lemma 2.6. \blacksquare

Next we discuss estimates of weak solutions of (1.1) in parabolic cylinders.

Theorem 4.2 *Under the assumption (H), let $u \in V_2(Q_T)$ be a weak solution of (1.1) in Q_T and $u = 0$ on $\partial_p Q_R$. Suppose that there exist λ and γ such that $\lambda < \gamma < Q+2$ and the function $\frac{r^{\gamma-\lambda}}{\varphi^2(r)}$ is almost increasing. Then $Xu \in L_\varphi^{2,\lambda}(Q_\rho)$. Furthermore, for any $\rho \leq R$, $Q_R \subset Q_T$, it follows*

$$\iint_{Q_\rho} |Xu|^2 dz \leq c \frac{\rho^\lambda \varphi^2(\rho)}{R^\lambda \varphi^2(R)} \iint_{Q_R} |Xu|^2 dz + c \rho^\lambda \varphi^2(\rho) (\|f\|_{L_\varphi^{2,\lambda}}^2 + \|g\|_{L_\varphi^{2,\lambda}}^2).$$

Proof. Let w be a weak solution to the system

$$\begin{cases} w_t^i + X_\alpha^*(a_{ij}^{\alpha\beta} X_\beta w^j) = 0, & \text{in } Q_R, \\ w = u, & \text{on } \partial_p Q_R, \end{cases} \quad (4.4)$$

Then $v = u - w$ satisfies

$$\begin{cases} v_t^i + X_\alpha^*(a_{ij}^{\alpha\beta} X_\beta v^j) = g_i + X_\alpha^* f_i^\alpha, & \text{in } Q_R, \\ v = 0, & \text{on } \partial_p Q_R. \end{cases} \quad (4.5)$$

Multiplying both sides of the system in (4.5) by v^i and integrating on Q_R ,

$$\iint_{Q_R} (v_t^i v^i + a_{ij}^{\alpha\beta} X_\beta v^j X_\alpha v^i) dz = \iint_{Q_R} (g_i v^i + f_i^\alpha X_\alpha v^i) dz.$$

Using (3.18),

$$\begin{aligned} &\iint_{Q_R} (v_t^i v^i + a_{ij}^{\alpha\beta} X_\beta v^j X_\alpha v^i) dz \\ &\leq C_\varepsilon \iint_{Q_R} (|g|^2 + |f|^2) dz + \varepsilon \iint_{Q_R} |v|^2 dz + \varepsilon \iint_{Q_R} |Xv|^2 dz \\ &\leq 2\varepsilon \iint_{Q_R} |Xv|^2 dz + C_\varepsilon \iint_{Q_R} (|g|^2 + |f|^2) dz. \end{aligned}$$

Since $\iint_{Q_R} v_t^i v^i dz = \int_{B_R} dx \int_{t_0-R^2}^{t_0} v^i dv^i \geq 0$ and (1.2), it yields

$$\iint_{Q_R} |Xv|^2 dz \leq C_\varepsilon \iint_{Q_R} (|g|^2 + |f|^2) dz. \quad (4.6)$$

Thanks to Theorem 4.1, we have

$$\begin{aligned} \iint_{Q_\rho} |Xu|^2 dz &\leq 2 \iint_{Q_\rho} |Xw|^2 dz + 2 \iint_{Q_\rho} |Xv|^2 dz \\ &\leq c\left(\frac{\rho}{R}\right)^\mu \iint_{Q_R} |Xw|^2 dz + c \iint_{Q_R} |Xv|^2 dz \\ &\leq c\left(\frac{\rho}{R}\right)^\mu \iint_{Q_R} |Xu|^2 dz + c \iint_{Q_R} |Xv|^2 dz \\ &\leq c\left(\frac{\rho}{R}\right)^\mu \iint_{Q_R} |Xu|^2 dz + C_\varepsilon \iint_{Q_R} (|g|^2 + |f|^2) dz \\ &\leq c\left(\frac{\rho}{R}\right)^\mu \iint_{Q_R} |Xu|^2 dz + c\varphi^2(R)R^\lambda (\|f\|_{L_\varphi^{2,\lambda}}^2 + \|g\|_{L_\varphi^{2,\lambda}}^2). \end{aligned}$$

Now letting $H(\rho) = \iint_{Q_\rho} |Xu|^2 dz$, $H(R) = \iint_{Q_R} |Xu|^2 dz$, $B = \|f\|_{L_\varphi^{2,\lambda}}^2 + \|g\|_{L_\varphi^{2,\lambda}}^2$, $F(R) = \varphi^2(R)R^\lambda$ and $\beta = Q + 2$, and noting that the function $\frac{\rho^\gamma}{F(\rho)}$ is almost increasing in $(0, R_0]$, we have by Lemma 2.6 that

$$\iint_{Q_\rho} |Xu|^2 dz \leq c \frac{\rho^\lambda \varphi^2(\rho)}{R^\lambda \varphi^2(R)} \iint_{Q_R} |Xu|^2 dz + c \rho^\lambda \varphi^2(\rho) (\|f\|_{L_\varphi^{2,\lambda}}^2 + \|g\|_{L_\varphi^{2,\lambda}}^2).$$

This proof is completed. ■

Proof for Theorem 1.1. By Theorem 4.2 and the cutoff function technique, it is easy to see that Theorem 1.1 is true, and we omit the details. ■

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Department of Applied mathematics, Northwestern polytechnical university; Key laboratory of Space Applied physics and chemistry, Ministry of Education, Xi'an, shaanxi, 710129, China

E-mail addresses: dongyan8617@sina.com

E-mail addresses: zhumaochun2006@126.com

E-mail addresses: pengchengniu@nwpu.edu.cn